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BROWDER'S TYPE CONVERGENCE THEOREMS FOR ONE-PARAMETER SEMIGROUPS OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

BY

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ABSTRACT

In this paper, we prove Browder's type convergence theorems for one-parameter strongly continuous semigroups of nonexpansive mappings in Banach spaces.

1. Introduction

Let C be a closed convex subset of a Banach space E. A mapping T on C is called a nonexpansive mapping if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. We know that $F(T)$ is nonempty in the case that C is weakly compact and has normal structure; see Kirk [13]. See also [1, 5, 10] and others. Fix $u \in C$. Then for each $\alpha \in (0, 1)$, there exists a unique point x_α in C satisfying $x_\alpha = (1 - \alpha)Tx_\alpha + \alpha u$ because the mapping $x \mapsto (1 - \alpha)Tx + \alpha u$ is contractive; see [2]. In 1967, Browder [7] proved the following:

Theorem 1 (Browder [7]): *Let C be a closed convex subset of a Hilbert space E and let T be a nonexpansive mapping on C with a fixed point. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ converging to 0. Fix $u \in C$ and define a sequence $\{u_n\}$ in C by*

$$u_n = (1 - \alpha_n)Tu_n + \alpha_n u$$

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for $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to the element of $F(T)$ nearest to u.

Reich extended this theorem to uniformly smooth Banach spaces in [18].

A family of mappings $\{T(t) : t \geq 0\}$ is called a one-parameter strongly continuous semigroup of nonexpansive mappings on C if the following are satisfied:

(sg 1) for each $t \geq 0$, $T(t)$ is a nonexpansive mapping on C;

(sg 2) $T(s+t) = T(s) \circ T(t)$ for all $s, t \geq 0$;

(sg 3) for each $x \in C$, the mapping $t \mapsto T(t)x$ from $[0, \infty)$ into C is strongly continuous.

We denote by $F(\mathcal{T})$ the set of common fixed points of $\{T(t) : t \geq 0\}$. We know that $F(\mathcal{T})$ is nonempty if C is weakly compact and has the fixed point property for nonexpansive mappings; see Bruck [9]. See also [3, 6, 14] and others. Browder's type convergence theorems for one-parameter semigroups of nonexpansive mappings are proved in [21–25] and others. For example, the following theorem is proved in [25].

THEOREM 2 ([25]): *Let C be a closed convex subset of a Hilbert space E. Let $\{T(t) : t \geq 0\}$ be a one-parameter strongly continuous semigroup of nonexpansive mappings on C with $F(\mathcal{T}) \neq \varnothing$. Let τ be a nonnegative real number. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $0 < \tau + t_n$ and $t_n \neq 0$ for $n \in \mathbb{N}$, and $\lim_n t_n = \lim_n \alpha_n / t_n = 0$. Fix $u \in C$ and define a sequence $\{u_n\}$ in C by*

$$u_n = (1 - \alpha_n) T(\tau + t_n) u_n + \alpha_n u$$

for $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to the element of $F(\mathcal{T})$ nearest to u.

This theorem is a generalization of the result in [22]. In [22], we assumed that $\tau = 0$ and $T(0)$ is the identity mapping on C.

In this paper, we extend Theorem 2 to Banach spaces; see Theorems 3–5 in Section 4. Proofs of our results are more difficult than that of Theorem 2.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers, and by $\mathbb{R}$ the set of real numbers. For a real number t, we denote by $[t]$ the maximum integer not exceeding t. It is obvious that for positive real numbers p and q,

$$0 \leq p - [p/q]q < q$$

holds.

Let $\{x_n\}$ be a sequence in a topological space X. By the Axiom of Choice, there exist a directed set $(D, \leq)$ and a universal subnet $\{x_{f(\nu)} : \nu \in D\}$ of $\{x_n\}$, i.e.,

(i) f is a mapping from D into $\mathbb{N}$ such that for each $n \in \mathbb{N}$ there exists $\nu_0 \in D$ such that $\nu \geq \nu_0$ implies $f(\nu) \geq n$; and

(ii) for each subset A of X, there exists $\nu_0 \in D$ such that either $\{x_{f(\nu)} : \nu \geq \nu_0\} \subset A$ or $\{x_{f(\nu)} : \nu \geq \nu_0\} \subset X \setminus A$ holds.

In this paper, we use $\{x_\nu : \nu \in D\}$ instead of $\{x_{f(\nu)} : \nu \in D\}$, for short. We know that if a net $\{x_\nu\}$ is universal, then for every mapping g from X into an arbitrary set Y, $\{g(x_\nu)\}$ is also universal. We also know that if X is compact, then a universal net $\{x_\nu\}$ always converges. See [12] for details.

Let E be a real Banach space. We denote by E^* the dual of E. E is called uniformly convex if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\frac{\|x + y\|}{2} < 1 - \delta$$

for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$. E is said to be smooth or said to have a Gâteaux differentiable norm if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in E$ with $\|x\| = \|y\| = 1$. E is said to have a uniformly Gâteaux differentiable norm if for each $y \in E$ with $\|y\| = 1$, the limit is attained uniformly in $x \in E$ with $\|x\| = 1$. E is said to be uniformly smooth or said to have a uniformly Fréchet differentiable norm if the limit is attained uniformly in $x, y \in E$ with $\|x\| = \|y\| = 1$. E is said to have the Opial property [15] if for each weakly convergent sequence $\{x_n\}$ in E with weak limit x_0,

$$\liminf_{n \to \infty} \|x_n - x_0\| < \liminf_{n \to \infty} \|x_n - x\|$$

hold for all $x \in E$ with $x \neq x_0$. We remark that we may replace 'lim inf' by 'lim sup'. That is, E has the Opial property if and only if for each weakly convergent sequence $\{x_n\}$ in E with weak limit x_0,

$$\limsup_{n \to \infty} \|x_n - x_0\| < \limsup_{n \to \infty} \|x_n - x\|$$

hold for all $x \in E$ with $x \neq x_0$.

Let E be a smooth Banach space. The duality mapping J from E into E^* is defined by

$$\langle x, J(x) \rangle = \|x\|^2 = \|J(x)\|^2$$

for all $x \in E$. J is said to be weakly sequentially continuous at zero if for every sequence $\{x_n\}$ in E which converges weakly to $0 \in E$, $\{J(x_n)\}$ converges weakly* to $0 \in E^*$.

A convex subset C of a Banach space E is said to have normal structure [4] if for every bounded convex subset K of C which contains more than one point, there exists $z \in K$ such that

$$\sup_{x \in K} \|x - z\| < \sup_{x,y \in K} \|x - y\|.$$

We know that compact convex subsets of any Banach spaces, and closed convex subsets of uniformly convex Banach spaces have normal structure. Turett [28] proved that uniformly smooth Banach spaces have normal structure. Also, Gossez and Lami Dozo [11] proved that every weakly compact convex subset C of a Banach space with the Opial property has normal structure. We recall that a closed convex subset C of a Banach space E is said to have the fixed point property for nonexpansive mappings (FPP, for short) if for every bounded closed convex subset K of C, every nonexpansive self-mapping on K has a fixed point. So, by Kirk's fixed point theorem [13], every weakly compact convex subset with normal structure has FPP.

Let C and K be subsets of a Banach space E. A mapping P from C into K is called sunny [8] if

$$P(Px + t(x - Px)) = Px$$

for $x \in C$ with $Px + t(x - Px) \in C$ and $t \geq 0$. The following is proved in [16]; see also [26].

Lemma 1 (Reich [16]): *Let E be a smooth Banach space and let C be a convex subset of E. Let K be a subset of C and let P be a retraction from C onto K. Then the following are equivalent:*

(i) $\langle x - Px, J(Px - y)\rangle \geq 0$ *for all* $x \in C$ *and* $y \in K$;

(ii) P *is both sunny and nonexpansive.*

Hence, there is at most one sunny nonexpansive retraction from C onto K.

The following lemma is essentially proved in [17]; see also [27].

Lemma 2 (Reich [17]): *Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm. Let $\{x_\alpha : \alpha \in D\}$ be a net in E and let $z \in C$. Suppose that the limits of $\{\|x_\alpha - y\|\}$ exist for all $y \in C$. Then the following are equivalent:*

(i) $\lim_{\alpha \in D} \|x_\alpha - z\| = \min_{y \in C} \lim_{\alpha \in D} \|x_\alpha - y\|$;

(ii) $\limsup_{\alpha\in D}\langle y-z, J(x_\alpha - z)\rangle \le 0$ *for all* $y\in C$;
(iii) $\liminf_{\alpha\in D}\langle y-z, J(x_\alpha - z)\rangle \le 0$ *for all* $y\in C$.

Proof: Fix $y \in C$. Then there exists $\alpha_1 \in D$ such that

$$\|x_\beta - y\| \le \lim_{\alpha\in D}\|x_\alpha - y\| + 1$$

for all $\beta \in D$ with $\beta \ge \alpha_1$. Hence $\{x_\alpha : \alpha \ge \alpha_1\}$ is bounded. Thus, without loss of generality, we may assume that $\{x_\alpha\}$ is bounded. We first show (i) implies (ii). We assume that $\lim_\alpha \|x_\alpha - z\| = \min_{y\in C}\lim_\alpha \|x_\alpha - y\|$. For $y \in C$ and $t \in (0,1)$, we have

$$\begin{aligned}\|x_\alpha - z\|^2 &= \|x_\alpha - tz - (1-t)y + (1-t)(y-z)\|^2\\ &\ge \|x_\alpha - tz - (1-t)y\|^2 + 2(1-t)\langle y-z, J(x_\alpha - tz - (1-t)y)\rangle.\end{aligned}$$

Since the norm of E is uniformly Gâteaux differentiable, the duality mapping is uniformly continuous on bounded subsets of E from the strong topology of E into the weak* topology of E^*; see Lemma 2.2 in Reich [19]. Therefore, for each $\varepsilon > 0$, there exists $t \in (0,1)$ such that

$$|\langle y-z, J(x_\alpha - tz - (1-t)y) - J(x_\alpha - z)\rangle| < \varepsilon$$

for all $\alpha \in D$. So, we have

$$\begin{aligned}\langle y-z, J(x_\alpha - z)\rangle &< \varepsilon + \langle y-z, J(x_\alpha - tz-(1-t)y)\rangle\\ &\le \varepsilon + \frac{1}{2(1-t)}(\|x_\alpha - z\|^2 - \|x_\alpha - tz - (1-t)y\|^2)\end{aligned}$$

and hence

$$\begin{aligned}\limsup_{\alpha\in D}\langle y-z, J(x_\alpha - z)\rangle &\le \varepsilon + \frac{1}{2(1-t)}(\lim_{\alpha\in D}\|x_\alpha - z\|^2\\ &\qquad - \lim_{\alpha\in D}\|x_\alpha - tz-(1-t)y\|^2)\\ &\le \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain (ii). It is obvious that (ii) implies (iii). We shall prove (iii) implies (i). We assume that $\liminf_\alpha\langle y-z, J(x_\alpha - z)\rangle \le 0$ for all $y \in C$. Fix $y \in C$. From

$$\begin{aligned}\|x_\alpha - y\|^2 - \|x_\alpha - z\|^2 &\ge 2\langle z-y, J(x_\alpha - z)\rangle\\ &= -2\langle y-z, J(x_\alpha - z)\rangle,\end{aligned}$$

we have

$$\big(\lim_{\alpha\in D}\|x_\alpha - y\|\big)^2 - \big(\lim_{\alpha\in D}\|x_\alpha - z\|\big)^2 \geq -2\liminf_{\alpha\in D}\langle y-z, J(x_\alpha - z)\rangle$$
$$\geq 0.$$

Therefore

$$\lim_{\alpha\in D}\|x_\alpha - z\| \leq \lim_{\alpha\in D}\|x_\alpha - y\|.$$

This implies (i). ∎

3. Lemmas

In this section, we prove some lemmas, which are used in the proofs of our main results.

The following is essentially proved in [25].

LEMMA 3 ([25]): *Let C be a closed convex subset of a Banach space E. Let $\{T(t) : t \geq 0\}$ be a one-parameter strongly continuous semigroup of nonexpansive mappings on C. Let τ be a nonnegative real number. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $0 < \tau + t_n$ and $t_n \neq 0$ for $n \in \mathbb{N}$, and $\lim_n t_n = \lim_n \alpha_n / t_n = 0$. Fix $u \in C$ and define a sequence $\{u_n\}$ in C by*

$$u_n = (1-\alpha_n)T(\tau + t_n)u_n + \alpha_n u$$

for $n \in \mathbb{N}$. Then the following hold:

(i) *For $n \in \mathbb{N}$,*

$$\|T(0)u_n - u_n\| \leq 2\alpha_n \|T(\tau + t_n)u_n - u\|.$$

(ii) *For $n \in \mathbb{N}$ and $x \in C$,*

$$\|u_n - T(\tau)x\| \leq \alpha_n\|T(\tau + t_n)u_n - u\| + \|u_n - x\| + \|T(|t_n|)x - T(0)x\|.$$

(iii) *If $T(\tau)x = x$ for some $x \in C$, then $T(0)x = x$.*

(iv) *If $T(\tau)x = x$ for some $x \in C$ and $0 < t_n < t$, then*

$$\|u_n - T(t)x\| \leq \|u_n - T(0)u_n\| + (t\alpha_n/t_n)\|T(\tau + t_n)u_n - u\|$$
$$+ \|u_n - x\| + \max\{\|T(s)x - T(0)x\| : 0 \leq s \leq t_n\}.$$

(v) *If $T(\tau)x = x$ for some $x \in C$, $t_n < 0 < t < \tau$ and $-2t_n < \tau - t$, then*

$$\|u_n - T(t)x\| \leq \|u_n - T(0)u_n\| + (-(\tau - t)\alpha_n/t_n)\|T(\tau + t_n)u_n - u\|$$
$$+ \|u_n - x\| + \max\{\|T(s)x - T(0)x\| : 0 \leq s \leq -t_n\}.$$

Proof: It is obvious that

$$\|T(\tau + t_n)u_n - u_n\| = \alpha_n \|T(\tau + t_n)u_n - u\|$$

for $n \in \mathbb{N}$. We note that

$$\begin{aligned}\|T(t)x - T(t)y\| &= \|T(t+0)x - T(t+0)y\| \\ &= \|T(t) \circ T(0)x - T(t) \circ T(0)y\| \\ &\leq \|T(0)x - T(0)y\|\end{aligned}$$

for all $x, y \in C$ and $t \geq 0$. We have

$$\begin{aligned}\|T(0)u_n - u_n\| &\leq \|T(0)u_n - T(0 + \tau + t_n)u_n\| + \|T(\tau + t_n)u_n - u_n\| \\ &= \|T(0)u_n - T(0) \circ T(\tau + t_n)u_n\| + \|T(\tau + t_n)u_n - u_n\| \\ &\leq 2\|T(\tau + t_n)u_n - u_n\| \\ &= 2\alpha_n \|T(\tau + t_n)u_n - u\|\end{aligned}$$

for all $n \in \mathbb{N}$. This is (i). We have

$$\begin{aligned}\|u_n - T(\tau)x\| \leq &\|u_n - T(\tau + t_n)u_n\| \\ &+ \|T(\tau + t_n)u_n - T(\tau + t_n)x\| + \|T(\tau + t_n)x - T(\tau)x\| \\ \leq &\alpha_n \|T(\tau + t_n)u_n - u\| + \|u_n - x\| + \|T(|t_n|)x - T(0)x\|\end{aligned}$$

for all $n \in \mathbb{N}$ and $x \in C$. This is (ii). We shall prove (iii). Since $T(\tau)x = x$, we have

$$T(0)x = T(0) \circ T(\tau)x = T(0 + \tau)x = T(\tau)x = x.$$

We shall prove (iv). From the assumption, we have

$$\begin{aligned}\|u_n - T(t)x\| =&\|u_n - T(t) \circ T(\tau)^{[t/t_n]}x\| \\ =&\|u_n - T([t/t_n]\tau + t)x\| \\ \leq&\|u_n - T(0)u_n\| \\ &+ \sum_{k=0}^{[t/t_n]-1} \|T((k+1)(\tau + t_n))u_n - T(k(\tau + t_n))u_n\| \\ &+ \|T([t/t_n](\tau + t_n))u_n - T([t/t_n](\tau + t_n))x\| \\ &+ \|T([t/t_n](\tau + t_n))x - T([t/t_n]\tau + t)x\| \\ \leq&\|u_n - T(0)u_n\| + [t/t_n]\|T(\tau + t_n)u_n - u_n\| \\ &+ \|u_n - x\| + \|T([t/t_n]t_n)x - T(t)x\| \\ \leq&\|u_n - T(0)u_n\| + [t/t_n]\alpha_n \|T(\tau + t_n)u_n - u\|\end{aligned}$$

$$
\begin{aligned}
&+\|u_n - x\| + \|T(t-[t/t_n]t_n)x - T(0)x\| \\
\le&\|u_n - T(0)u_n\| + (t\alpha_n/t_n)\|T(\tau+t_n)u_n - u\| \\
&+\|u_n - x\| + \max\{\|T(s)x - T(0)x\| : 0 \le s \le t_n\}.
\end{aligned}
$$

Let us prove (v). We put $p_n = -t_n > 0$ for $n \in \mathbb{N}$. From the assumption, we have

$$
\begin{aligned}
\|u_n - T(t)x\| =&\|u_n - T(t)\circ T(\tau)^{[(\tau-t)/p_n]-1}x\| \\
=&\|u_n - T([(\tau-t)/p_n]\tau - \tau + t)x\| \\
\le&\|u_n - T(0)u_n\| \\
&+\sum_{k=0}^{[(\tau-t)/p_n]-1} \|T((k+1)(\tau+t_n))u_n - T(k(\tau+t_n))u_n\| \\
&+\|T([(\tau-t)/p_n](\tau+t_n))u_n - T([(\tau-t)/p_n](\tau+t_n))x\| \\
&+\|T([(\tau-t)/p_n](\tau+t_n))x - T([(\tau-t)/p_n]\tau - \tau + t)x\| \\
\le&\|u_n - T(0)u_n\| + [(\tau-t)/p_n]\|T(\tau+t_n)u_n - u_n\| \\
&+\|u_n - x\| + \|T(\tau + [(\tau-t)/p_n]t_n)x - T(t)x\| \\
=&\|u_n - T(0)u_n\| + [(\tau-t)/p_n]\alpha_n\|T(\tau+t_n)u_n - u\| \\
&+\|u_n - x\| + \|T(\tau - [(\tau-t)/p_n]p_n)x - T(t)x\| \\
\le&\|u_n - T(0)u_n\| + ((\tau-t)\alpha_n/p_n)\|T(\tau+t_n)u_n - u\| \\
&+\|u_n - x\| + \|T((\tau-t) - [(\tau-t)/p_n]p_n)x - T(0)x\| \\
\le&\|u_n - T(0)u_n\| + ((\tau-t)\alpha_n/p_n)\|T(\tau+t_n)u_n - u\| \\
&+\|u_n - x\| + \max\{\|T(s)x - T(0)x\| : 0 \le s \le p_n\} \\
=&\|u_n - T(0)u_n\| + (-(\tau-t)\alpha_n/t_n)\|T(\tau+t_n)u_n - u\| \\
&+\|u_n - x\| + \max\{\|T(s)x - T(0)x\| : 0 \le s \le -t_n\}.
\end{aligned}
$$

This completes the proof. ∎

Using Lemma 3, we prove the following useful lemma.

Lemma 4: *Let E, C, $\{T(t) : t \ge 0\}$, τ, $\{\alpha_n\}$, $\{t_n\}$, u and $\{u_n\}$ be as in Lemma 3. Assume that $\{u_n\}$ is bounded. Let $\{u_\nu : \nu \in D\}$ be a subnet of $\{u_n\}$. Then the following hold:*

(i) $\limsup_\nu \|u_\nu - T(\tau)x\| \le \limsup_\nu \|u_\nu - x\|$ *for all* $x \in C$.

(ii) $\limsup_\nu \|u_\nu - T(0)x\| \le \limsup_\nu \|u_\nu - x\|$ *for all* $x \in C$.

(iii) *If* $0 < t \le \tau$, *then* $\limsup_\nu \|u_\nu - T(t)x\| \le \limsup_\nu \|T(\tau - t)u_\nu - x\|$ *for all* $x \in C$.

(iv) *If* $T(\tau)x = x$ *for some* $x \in C$, *then* $\limsup_\nu \|u_\nu - T(t)x\| \le \limsup_\nu \|u_\nu - x\|$ *for all* $t \ge 0$.

Proof: From

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad T(\tau + t_n)u_n = \frac{1}{1-\alpha_n}u_n - \frac{\alpha_n}{1-\alpha_n}u$$

for $n \in \mathbb{N}$, we note that $\{T(\tau + t_n)u_n\}$ is bounded. By Lemma 3 (ii), we have

$$\begin{aligned}\limsup_{\nu\in D} \|u_\nu - T(\tau)x\| &\le \limsup_{\nu\in D}(\alpha_\nu \|T(\tau + t_\nu)u_\nu - u\| \\ &\qquad + \|u_\nu - x\| + \|T(|t_\nu|)x - T(0)x\|) \\ &= \limsup_{\nu\in D} \|u_\nu - x\|\end{aligned}$$

for all $x \in C$. This is (i). By Lemma 3 (i), we have

$$\lim_{n\to\infty} \|T(0)u_n - u_n\| \le \lim_{n\to\infty} 2\alpha_n \|T(\tau + t_n)u_n - u\| = 0.$$

So, we obtain

$$\begin{aligned}\limsup_{\nu\in D} \|u_\nu - T(0)x\| &\le \limsup_{\nu\in D}(\|u_\nu - T(0)u_\nu\| + \|T(0)u_\nu - T(0)x\|) \\ &\le \limsup_{\nu\in D}(\|u_\nu - T(0)u_\nu\| + \|u_\nu - x\|) \\ &= \limsup_{\nu\in D} \|u_\nu - x\|\end{aligned}$$

for all $x \in C$. This is (ii). For $x \in C$ and $t \in \mathbb{R}$ with $0 < t \le \tau$, we have

$$\begin{aligned}&\limsup_{\nu\in D} \|u_\nu - T(t)x\| \\ &\quad \le \limsup_{\nu\in D}(\|u_\nu - T(\tau + t_\nu)u_\nu\| + \|T(\tau + t_\nu)u_\nu - T(t + t_\nu)x\| \\ &\qquad\qquad + \|T(t + t_\nu)x - T(t)x\|) \\ &\quad = \limsup_{\nu\in D}(\alpha_\nu \|T(\tau + t_\nu)u_\nu - u\| + \|T(t + t_\nu) \circ T(\tau - t)u_\nu - T(t + t_\nu)x\| \\ &\qquad\qquad + \|T(t + t_\nu)x - T(t)x\|) \\ &\quad \le \limsup_{\nu\in D}(\alpha_\nu \|T(\tau + t_\nu)u_\nu - u\| + \|T(\tau - t)u_\nu - x\| \\ &\qquad\qquad + \|T(|t_\nu|)x - T(0)x\|) \\ &\quad = \limsup_{\nu\in D} \|T(\tau - t)u_\nu - x\|.\end{aligned}$$

This is (iii). Fix $x \in C$ with $T(\tau)x = x$. By the assumption, it is obvious that

$$\limsup_{\nu\in D} \|u_\nu - T(\tau)x\| = \limsup_{\nu\in D} \|u_\nu - x\|.$$

By Lemma 3 (iii), we have

$$\limsup_{\nu\in D}\|u_\nu - T(0)x\| = \limsup_{\nu\in D}\|u_\nu - x\|.$$

Fix $t\in\mathbb{R}$ with $t>0$ and $t\neq\tau$. In the case of $\tau=0$, by Lemma 3 (i) and (iv), we obtain

$$\begin{aligned}\limsup_{\nu\in D}&\|u_\nu - T(t)x\|\\ &\le\limsup_{\nu\in D}(\|u_\nu - T(0)u_\nu\| + (t\alpha_\nu/t_\nu)\|T(\tau+t_\nu)u_\nu - u\|\\ &\qquad + \|u_\nu - x\| + \max\{\|T(s)x - T(0)x\| : 0\le s\le t_\nu\})\\ &=\limsup_{\nu\in D}\|u_\nu - x\|.\end{aligned}$$

In the case of $0<t<\tau$, by Lemma 3 (iv) and (v), we have

$$\begin{aligned}\|u_\nu - T(t)x\| \le&\|u_\nu - T(0)u_\nu\| + \max\{t,\tau-t\}|\alpha_\nu/t_\nu|\|T(\tau+t_\nu)u_\nu - u\|\\ &+ \|u_\nu - x\| + \max\{\|T(s)x - T(0)x\| : 0\le s\le |t_\nu|\}\\ \le&\|u_\nu - T(0)u_\nu\| + |\tau\alpha_\nu/t_\nu|\|T(\tau+t_\nu)u_\nu - u\|\\ &+ \|u_\nu - x\| + \max\{\|T(s)x - T(0)x\| : 0\le s\le |t_\nu|\}\end{aligned}$$

for large $\nu\in D$. Thus, using Lemma 3 (i), we obtain

$$\begin{aligned}\limsup_{\nu\in D}&\|u_\nu - T(t)x\|\\ &\le\limsup_{\nu\in D}(\|u_\nu - T(0)u_\nu\| + |\tau\alpha_\nu/t_\nu|\|T(\tau+t_\nu)u_\nu - u\|\\ &\qquad + \|u_\nu - x\| + \max\{\|T(s)x - T(0)x\| : 0\le s\le |t_\nu|\})\\ &=\limsup_{\nu\in D}\|u_\nu - x\|.\end{aligned}$$

In the case of $0<\tau<t$, we have

$$\begin{aligned}T(t)x &= T(t-[t/\tau]\tau+[t/\tau]\tau)x = T(t-[t/\tau]\tau)\circ T(\tau)^{[t/\tau]}x\\ &= T(t-[t/\tau]\tau)x.\end{aligned}$$

Hence, we obtain

$$\limsup_{\nu\in D}\|u_\nu - T(t)x\| = \limsup_{\nu\in D}\|u_\nu - T(t-[t/\tau]\tau)x\| \le \limsup_{\nu\in D}\|u_\nu - x\|$$

because $0\le t-[t/\tau]\tau<\tau$. This completes the proof. ∎

We continue to prove lemmas.

Lemma 5: *Let E, C, $\{T(t) : t \geq 0\}$, τ, $\{\alpha_n\}$, $\{t_n\}$, u and $\{u_n\}$ be as in Lemma 3. Assume that $\{u_n\}$ converges strongly to some point $x \in C$. Then x is a common fixed point of $\{T(t) : t \geq 0\}$.*

Proof: We note that $\{u_n\}$ is bounded because $\{u_n\}$ converges. From Lemma 4 (i), we have

$$\limsup_{n\to\infty} \|u_n - T(\tau)x\| \leq \lim_{n\to\infty} \|u_n - x\| = 0$$

and hence $\{u_n\}$ converges to $T(\tau)x$. Therefore $T(\tau)x = x$ by the assumption. For every $t \geq 0$, from Lemma 4 (iv), we have

$$\limsup_{n\to\infty} \|u_n - T(t)x\| \leq \lim_{n\to\infty} \|u_n - x\| = 0$$

and hence $\{u_n\}$ converges to $T(t)x$. Therefore $T(t)x = x$ for all $t \geq 0$. This completes the proof. ∎

Lemma 6: *Let E, C, $\{T(t) : t \geq 0\}$, τ, $\{\alpha_n\}$, $\{t_n\}$, u and $\{u_n\}$ be as in Lemma 3. Assume that E is smooth and $z \in C$ is a common fixed point of $\{T(t) : t \geq 0\}$. Then*

$$\langle u_n - u, J(u_n - z)\rangle \leq 0$$

for all $n \in \mathbb{N}$.

Proof: We have

$$\begin{aligned}
\frac{\alpha_n}{1-\alpha_n}\langle u_n - u, J(u_n - z)\rangle &= \langle T(\tau + t_n)u_n - u_n, J(u_n - z)\rangle \\
&= \langle T(\tau + t_n)u_n - z, J(u_n - z)\rangle + \langle z - u_n, J(u_n - z)\rangle \\
&= \langle T(\tau + t_n)u_n - z, J(u_n - z)\rangle - \|u_n - z\|^2 \\
&\leq \|T(\tau + t_n)u_n - z\|\|u_n - z\| - \|u_n - z\|^2 \\
&\leq \|u_n - z\|^2 - \|u_n - z\|^2 \\
&= 0.
\end{aligned}$$

Hence we obtain

$$\langle u_n - u, J(u_n - z)\rangle \leq 0$$

for all $n \in \mathbb{N}$. ∎

LEMMA 7: *Let E, C, $\{T(t) : t \geq 0\}$, τ, $\{\alpha_n\}$, $\{t_n\}$, u and $\{u_n\}$ be as in Lemma 3. Assume that E is smooth. Then $\{u_n\}$ has at most one cluster point.*

Proof: We assume that a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ converges strongly to x, and that another subsequence $\{u_{n_j}\}$ of $\{u_n\}$ converges strongly to y. By Lemma 5, x and y are common fixed points of $\{T(t) : t \geq 0\}$. So, by Lemma 6, we have

$$\langle u_{n_i} - u, J(u_{n_i} - y)\rangle \leq 0$$

for all $i \in \mathbb{N}$. Therefore we obtain

$$\langle x - u, J(x - y)\rangle \leq 0.$$

Similarly we can prove

$$\langle y - u, J(y - x)\rangle \leq 0.$$

So, we obtain

$$\begin{aligned}\|x - y\|^2 &= \langle x - y, J(x - y)\rangle \\ &= \langle x - u, J(x - y)\rangle + \langle u - y, J(x - y)\rangle \\ &= \langle x - u, J(x - y)\rangle + \langle y - u, J(y - x)\rangle \\ &\leq 0.\end{aligned}$$

This implies $x = y$. This completes the proof. ∎

LEMMA 8: *Let E be a reflexive Banach space with uniformly Gâteaux differentiable norm and let C be a closed convex subset of E with the fixed point property for nonexpansive mappings. Let $\{T(t) : t \geq 0\}$, τ, $\{\alpha_n\}$, $\{t_n\}$, u and $\{u_n\}$ be as in Lemma 3. Assume that $\{T(t) : t \geq 0\}$ has a common fixed point. Then $\{u_n\}$ has a cluster point.*

Remark: Our proof employs the method in the proof of Theorem 2 in Reich [20].

Proof: Fix a common fixed point w of $\{T(t) : t \geq 0\}$. Since

$$\begin{aligned}\|u_n - w\| &= \|(1 - \alpha_n)T(\tau + t_n)u_n + \alpha_n u - w\| \\ &\leq (1 - \alpha_n)\|T(\tau + t_n)u_n - w\| + \alpha_n\|u - w\| \\ &\leq (1 - \alpha_n)\|u_n - w\| + \alpha_n\|u - w\|,\end{aligned}$$

we have $\|u_n - w\| \leq \|u - w\|$ for $n \in \mathbb{N}$. Therefore $\{u_n\}$ is bounded. Since

$$\|T(t)u_n - w\| \leq \|u_n - w\|$$

for all $t \in [0, \infty)$ and $n \in \mathbb{N}$, we have $\{T(t)u_n : t \in [0, \infty), n \in \mathbb{N}\}$ is bounded. Take a universal subnet $\{u_\nu : \nu \in D\}$ of $\{u_n\}$. Define two continuous convex functions f and g from C into $[0, \infty)$ by

$$f(x) = \sup_{s \in [0,\infty)} \lim_{\nu \in D} \|T(s)u_\nu - x\| \quad \text{and} \quad g(x) = \lim_{\nu \in D} \|u_\nu - x\|$$

for all $x \in C$. We note that g is well-defined because $\{\|u_\nu - x\|\}$ is a universal net in some compact subset of $\mathbb{R}$ for each $x \in C$. f is also well-defined because

$$\begin{aligned} \|T(s)u_\nu - x\| &\leq \|T(s)u_\nu - w\| + \|w - x\| \\ &\leq \|u_\nu - w\| + \|w - x\| \end{aligned}$$

for all $x \in X$, $s \in [0, \infty)$ and $\nu \in D$. From the reflexivity of E and $\lim_{\|x\| \to \infty} g(x) = \infty$, we can put $r = \min_{x \in C} g(x)$ and define a nonempty weakly compact convex subset A of C by

$$A = \{x \in C : g(x) = r\}.$$

We shall prove that $\{T(t) : t \geq 0\}$ has a common fixed point in A. For each $x \in A$, by Lemma 4 (i), we have

$$r \leq g(T(\tau)x) = \lim_{\nu \in D} \|u_\nu - T(\tau)x\| \leq \lim_{\nu \in D} \|u_\nu - x\| = g(x) = r.$$

Hence A is $T(\tau)$-invariant. So, by the hypothesis, there exists a fixed point $y \in A$ of $T(\tau)$. We note that $T(t)y \in A$ for every $t \geq 0$, because

$$r \leq g(T(t)y) = \lim_{\nu \in D} \|u_\nu - T(t)y\| \leq \lim_{\nu \in D} \|u_\nu - y\| = g(y) = r$$

by Lemma 4 (iv). In the case of $\tau = 0$, we fix $x \in A$ and $t \geq 0$. Then we have

$$T(\tau) \circ T(0)x = T(\tau + 0)x = T(0)x$$

and hence $T(0)x$ is a fixed point of $T(\tau)$. So,

$$T(t)x = T(t + 0)x = T(t) \circ T(0)x \in A.$$

Therefore A is $T(t)$-invariant for every $t \geq 0$. By the hypothesis, there exists a common fixed point of $\{T(t) : t \geq 0\}$ in A. In the case of $\tau > 0$, we define a weakly compact convex subset B of A by

$$B = \{x \in C : f(x) \leq r, g(x) = r\}.$$

Since $T(s)y \in A$ for every $s \geq 0$, we have

$$\begin{aligned}\lim_{\nu \in D} \|T(s)u_\nu - y\| &= \lim_{\nu \in D} \|T(s)u_\nu - T(\tau)^{[s/\tau]+1} y\| \\ &= \lim_{\nu \in D} \|T(s)u_\nu - T(([s/\tau]+1)\tau)y\| \\ &\leq \lim_{\nu \in D} \|u_\nu - T(([s/\tau]+1)\tau - s)y\| \\ &= g(T(([s/\tau]+1)\tau - s)y) \\ &= r\end{aligned}$$

for every $s \geq 0$. So, we have $f(y) \leq r$ and hence $y \in B$. Therefore B is nonempty. We next show B is $T(t)$-invariant for every $t \geq 0$. Fix $x \in B$. By Lemma 4 (ii), we have

$$r \leq g(T(0)x) = \lim_{\nu \in D} \|u_\nu - T(0)x\| \leq \lim_{\nu \in D} \|u_\nu - x\| = g(x) = r.$$

We also have

$$\lim_{\nu \in D} \|T(s)u_\nu - T(0)x\| \leq \lim_{\nu \in D} \|T(s)u_\nu - x\| \leq f(x) \leq r$$

for all $s \geq 0$. Thus, $f(T(0)x) \leq r$ and hence $T(0)x \in B$. Fix $t \in \mathbb{R}$ with $0 < t \leq \tau$. Then we have

$$r \leq g(T(t)x) = \lim_{\nu \in D} \|u_\nu - T(t)x\| \leq \lim_{\nu \in D} \|T(\tau - t)u_\nu - x\| \leq f(x) \leq r$$

by Lemma 4 (iii). For $s \in \mathbb{R}$ with $0 \leq s < t$, since $0 < t - s \leq \tau$, we have

$$\begin{aligned}\lim_{\nu \in D} \|T(s)u_\nu - T(t)x\| &\leq \lim_{\nu \in D} \|u_\nu - T(t-s)x\| \\ &\leq \lim_{\nu \in D} \|T(\tau - t + s)u_\nu - x\| \\ &\leq f(x) \leq r\end{aligned}$$

by Lemma 4 (iii). For $s \in \mathbb{R}$ with $t \leq s$, we have

$$\lim_{\nu \in D} \|T(s)u_\nu - T(t)x\| \leq \lim_{\nu \in D} \|T(s-t)u_\nu - x\| \leq f(x) \leq r.$$

So, we obtain $f(T(t)x) \leq r$. Therefore $T(t)x \in B$. Fix $t \in \mathbb{R}$ with $\tau \leq t$. From $T(\tau)x \in B$, we have $T(\tau)^2 x \in B$ and hence $T(\tau)^n x \in B$ for $n \in \mathbb{N}$. So, we obtain

$$T(t)x = T(t - [t/\tau]\tau) \circ T(\tau)^{[t/\tau]} x \in B.$$

Therefore we have shown that B is $T(t)$-invariant for every $t \geq 0$. By the hypothesis, there exists a common fixed point of $\{T(t) : t \geq 0\}$ in B. Since

$B \subset A$, there exists $z \in A$ such that $T(t)z = z$ for all $t \geq 0$ in both cases. We shall prove that z is a cluster point of $\{u_n\}$. By Lemma 6, we have

$$\langle u_\nu - u, J(u_\nu - z)\rangle \leq 0$$

for all $\nu \in D$. On the other hand, from $z \in A$, we have

$$\lim_{\nu \in D} \langle u - z, J(u_\nu - z)\rangle \leq 0$$

by Lemma 2. Hence

$$\lim_{\nu \in D} \|u_\nu - z\|^2 = \lim_{\nu \in D} \langle u_\nu - z, J(u_\nu - z)\rangle \leq 0$$

holds. Therefore

$$\liminf_{n\to\infty} \|u_n - z\| \leq \lim_{\nu \in D} \|u_\nu - z\| = 0,$$

that is, z is a cluster point of $\{u_n\}$. This completes the proof. ∎

Lemma 9: *Let E, C, $\{T(t) : t \geq 0\}$, τ, $\{\alpha_n\}$, and $\{t_n\}$ be as in Lemma 3. Assume that E is smooth. For each $u \in C$, define a sequence $\{Q(u,n)\}$ in C by*

$$Q(u,n) = (1-\alpha_n)T(\tau + t_n)Q(u,n) + \alpha_n u$$

for $n \in \mathbb{N}$. Suppose that $\{Q(u,n)\}$ converges strongly for every $u \in C$. Then

$$Pu = \lim_{n\to\infty} Q(u,n)$$

holds for every $u \in C$, where P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{T})$.

Proof: Define a mapping P from C into $F(\mathcal{T})$ by $Pu = \lim_n Q(u,n)$ for $u \in C$. We shall prove that such P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{T})$. By Lemma 5, we note that $Px \in F(\mathcal{T})$ for all $x \in C$. For $z \in F(\mathcal{T})$, since

$$z = (1-\alpha_n)T(\tau + t_n)z + \alpha_n z$$

for all $n \in \mathbb{N}$, we have $Q(z,n) = z$ for all $n \in \mathbb{N}$. Hence, we obtain $Pz = z$. Therefore we have shown that $P^2 = P$, i.e., P is a retraction from C onto $F(\mathcal{T})$. Fix $x \in C$ and $y \in F(\mathcal{T})$. Then from Lemma 6, we have

$$\langle Q(x,n) - x, J(Q(x,n) - y)\rangle \leq 0$$

for all $n \in \mathbb{N}$. Since $\{Q(x,n)\}$ converges strongly to Px, we obtain

$$\langle Px - x, J(Px - y)\rangle \leq 0.$$

So, by Lemma 1, such a mapping P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{T})$. This completes the proof. ∎

4. Main results

In this section, we prove our main results. We put $F(\mathcal{T}) = \bigcap_{t\geq 0} F(T(t))$.

Theorem 3: *Let E be a reflexive Banach space with uniformly Gâteaux differentiable norm and let C be a closed convex subset of E with the fixed point property for nonexpansive mappings. Let $\{T(t) : t \geq 0\}$, τ, $\{\alpha_n\}$, $\{t_n\}$, u and $\{u_n\}$ be as in Lemma 3. Assume that $F(\mathcal{T})$ is nonempty. Then $\{u_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{T})$.*

Proof: By Lemma 8, $\{u_n\}$ has a cluster point $z \in C$. Let $\{u_{n_k}\}$ be an arbitrary subsequence of $\{u_n\}$. By Lemma 8 again, $\{u_{n_k}\}$ has a cluster point $y \in C$, which is also a cluster point of $\{u_n\}$. So, by Lemma 7, we obtain $y = z$. Hence, there exists a subsequence $\{u_{n_{k_j}}\}$ of $\{u_{n_k}\}$ converging strongly to z. Since $\{u_{n_k}\}$ is arbitrary, we obtain that $\{u_n\}$ converges strongly to $z \in C$. So, by Lemma 9, we obtain the desired result. ∎

Theorem 4: *Let E be a smooth reflexive Banach space with the Opial property and let C be a closed convex subset of E. Assume that the duality mapping J of E is weakly sequentially continuous at zero. Let $\{T(t) : t \geq 0\}$, τ, $\{\alpha_n\}$, $\{t_n\}$, u and $\{u_n\}$ be as in Lemma 3. Assume that $F(\mathcal{T})$ is nonempty. Then $\{u_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{T})$.*

Remark: We may replace the condition of the reflexivity of E by the weaker condition that C is locally weakly compact.

Proof: From the proof of Lemma 8, we have that $\{u_n\}$ is bounded. Let $\{u_{n_k}\}$ be an arbitrary subsequence of $\{u_n\}$. Since E is reflexive, there exists a subsequence $\{u_{n_{k_j}}\}$ of $\{u_{n_k}\}$ which converges weakly to some point $z \in C$. We put $z_j = u_{n_{k_j}}$, $\beta_j = \alpha_{n_{k_j}}$ and $s_j = t_{n_{k_j}}$ for $j \in \mathbb{N}$. By Lemma 4 (i), we have

$$\limsup_{j\to\infty} \|z_j - T(\tau)z\| \leq \limsup_{j\to\infty} \|z_j - z\|.$$

Since E has the Opial property, we obtain $T(\tau)z = z$. By Lemma 4 (iv), we have

$$\limsup_{j\to\infty} \|z_j - T(t)z\| \leq \limsup_{j\to\infty} \|z_j - z\|$$

for all $t \geq 0$. Therefore z is a common fixed point of $\{T(t) : t \geq 0\}$. Using Lemma 6, we have

$$\begin{aligned} \|z_j - z\|^2 &= \langle z_j - z, J(z_j - z)\rangle \\ &= \langle z_j - u, J(z_j - z)\rangle + \langle u - z, J(z_j - z)\rangle \\ &\leq \langle u - z, J(z_j - z)\rangle \end{aligned}$$

for all $j \in \mathbb{N}$. Since J is weakly sequentially continuous at zero, we obtain that $\{z_j\}$ converges strongly to z. By Lemma 7, we know that $\{u_n\}$ has at most one cluster point. So, since $\{u_{n_k}\}$ is an arbitrary subsequence of $\{u_n\}$, we have that $\{u_n\}$ itself converges strongly to z. So, by Lemma 9, we obtain the desired result. ∎

By Theorems 3 and 4, we obtain the following.

Theorem 5: *Let C be a weakly compact convex subset of a Banach space E. Assume that either of the following holds:*

- *E is uniformly convex with uniformly Gâteaux differentiable norm;*
- *E is uniformly smooth; or*
- *E is a smooth Banach space with the Opial property and the duality mapping J of E is weakly sequentially continuous at zero.*

Let $\{T(t) : t \geq 0\}$ be a one-parameter strongly continuous semigroup of nonexpansive mappings on C. Let τ be a nonnegative real number. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $0 < \tau + t_n$ and $t_n \neq 0$ for $n \in \mathbb{N}$, and $\lim_n t_n = \lim_n \alpha_n/t_n = 0$. Fix $u \in C$ and define a sequence $\{u_n\}$ in C by

$$u_n = (1 - \alpha_n)T(\tau + t_n)u_n + \alpha_n u$$

for $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{T})$.

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